The Fermionic Path Integral

Grassman Calculus

Fermions anti-commute $\Rightarrow$ need a notion of numbers that satisfy $\{\psi(x), \chi(y)\} = 0$

A Grassman algebra is a set of objects G generated by a basis $\{\theta_i\}$

The θ_i are Grassman numbers.

They anti-commute $\theta_i\theta_j = -\theta_j\theta_i$

They add $\theta_i + \theta_j = \theta_j + \theta_i$

Can be multiplied by complex numbers $a\theta \in G$ for $\theta \in G$ and $a \in \mathbb{C}$

There is a zero element $\theta_i + 0 = \theta_i$

Given a single θ, the most general element is

$$g = a + b\,\theta \quad \text{w/} \quad a, b \in \mathbb{C} \qquad \text{since } \theta^2 = 0$$

For two θ_i's, $g = a + b\theta_1 + c\theta_2 + d\,\theta_1\theta_2$

Elements w/ even number of θ's commute
$\Rightarrow$ even-graded or bosinic subalgebra

Elements w/ odd number of θ's anti-commute

$\Rightarrow$ odd-graded or fermionic sub-algebra.

The fermionic sub-algebra does not close since $\theta_1 \theta_2$ is bosonic.

We will identify $\theta_1 = \psi(x_1)$, $\theta_2 = \psi(x_2)$, ...

so there are an infinite number of Grassmann numbers.

Lagrangian terms are bosonic.

To do path integral, we need to integrate $\mathcal{D}\psi$
$\Rightarrow$ need to define Grassmann integration

Want integration to be linear:

$$\int d\theta_1 \cdots d\theta_n \,(s X + t Y)$$

$$= s\int d\theta_1 \cdots d\theta_n \, X + t\int d\theta_1 \cdots d\theta_n \, Y \quad , \quad s,t \in \mathbb{C} \quad , \quad X, Y \in G$$

Do not need limits of integration since only one Grassmann number in each direction. Analog w/ $\int_{-\infty}^{\infty} dx\, f(x)$ bosonic ints.

Want int to act like summing $d\theta \Rightarrow d\theta$ is anti-commuting

Take case of single θ: $\int d\theta(a+b\theta) = a\int d\theta + b\int d\theta\,\theta$

Want int to map from $G \to \mathbb{C} \Rightarrow \int d\theta = 0$ and $\int d\theta\,\theta = 1$ by convention.

$\Rightarrow \int d\theta (a+b\theta) = b$

Derivatives: $\frac{d}{d\theta}(a+b\theta) = b \Rightarrow$ int and dif do same thing.

Multiple θ's $\Rightarrow$

$$\int d\theta_1 \dots d\theta_n \,\underline{X} = \frac{\partial}{\partial\theta_1} \circ\circ\circ \frac{\partial}{\partial\theta_n} \underline{X}$$

$\Rightarrow \int d\theta_1 \dots d\theta_n \; \theta_n \dots \theta_1 = 1$ where we eval nested ints from inside out.

$$\Rightarrow \int d\theta_1 d\theta_2 \,\theta_2\theta_1 = -\int d\theta_1 d\theta_2 \;\theta_1\theta_2 = 1$$

Note analog w/ $\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x+a)$

$\int d\theta\left(A + B\theta\right) = \int d\theta\left(A + B(\theta + \underline{X})\right)$ w/ $\underline{X} \in G$ w/ $\frac{\partial}{\partial\theta}\underline{X} = 0$

We will need Gaussian ints: for two θ_i's,

$$\int d\theta_1 d\theta_2 \exp\left(-\theta_1 A_{12}\theta_2\right) = \int d\theta_1 d\theta_2 \left(1 - A_{12}\theta_1\theta_2\right) = A_{12}$$

$\uparrow$ Taylor expansion (note no notion of small θ)

Next, take n θ_i's and n $\bar\theta_i$'s

$$\int d\bar\theta_1 \dots d\bar\theta_n d\theta_n \dots d\theta_1 \exp\left(-\bar\theta_i A_{ij}\theta_j\right)$$

$$= \int d\bar\theta_1 \dots d\bar\theta_n d\theta_n \dots d\theta_1 \left(1 - \bar\theta_i A_{ij}\theta_j + \tfrac{1}{2}(\bar\theta_i A_{ij}\theta_j)(\bar\theta_k A_{kl}\theta_l) + \dots\right)$$

Only term that survives is one w/ n θ_i + n $\bar\theta_i$

$$\Rightarrow \int d\bar\theta_1 \dots d\bar\theta_n d\theta_n \dots d\theta_1 \exp\left(-\bar\theta_i A_{ij}\theta_j\right) = \frac{1}{n!}\sum_{\substack{\text{perms}\\ \{i_n\}}} \pm A_{i_1 i_2} \cdots A_{i_{n-1} i_n}$$

If A_{ij} is a matrix, this is sum over elements $\{i,j\}$ where we choose each row and column once.

$\Rightarrow$ $n!$ from perms cancels $1/n!$

$$\Rightarrow \int d\bar\theta_1 \dots d\bar\theta_n d\theta_n \dots d\theta_1 \exp(-\bar\theta_i A_{ij} \theta_j) = \det(A)$$

Note difference from bosonic Gaussian integral:

$$\int dx_1 \dots dx_n \exp\left(-\tfrac{1}{2} x_i A_{ij} x_j\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}}$$

Finally, it is straight forward to generalize to include currents η_i + $\bar\eta_i$

$$\Rightarrow \int d\bar\theta_1 \dots d\bar\theta_n d\theta_n \dots d\theta_1 \exp\left(-\bar\theta_i A_{ij}\theta_j + \bar\eta_i \theta_i + \bar\theta_i \eta_i\right)$$

$$= e^{\vec{\bar\eta} A^{-1} \vec\eta} \int d\vec{\bar\theta}\, d\vec\theta \exp\left(-\vec{\bar\theta} - \vec{\bar\eta} A^{-1}\right) A\left(\vec\theta - A^{-1}\vec\eta\right) = \det(A) e^{\vec{\bar\eta} A^{-1}\vec\eta}$$

Path integral

Discrete $\to$ continuous : $i \to x$, $\theta_i \to \psi(x)$, $\bar\theta_i \to \bar\psi(x)$

$$\Rightarrow Z[\bar\eta, \eta] = \int \mathcal{D}[\bar\psi(x)] \mathcal{D}[\psi(x)] \exp\left[i \int d^4x \left(\bar\psi(i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta + i\varepsilon\bar\psi\psi\right)\right]$$

where $i\varepsilon$ comes from boundary condition @ $t = \pm\infty$

$$\Rightarrow Z[\bar\eta, \eta] = N \exp\left(i \int d^4x \int d^4y\, \bar\eta(y) \left(i\not\partial - m + i\varepsilon\right)^{-1} \eta(x)\right)$$

w/ $N = \det(i\not\partial - m)$ is an infinite constant

$$\Rightarrow \langle 0 | T\{\psi(x)\bar\psi(y)\} | 0 \rangle = \frac{1}{Z[0]} \frac{\partial^2}{\partial\bar\eta(x)\partial\eta(y)} Z[\bar\eta, \eta]\Big|_{\eta=0} = \frac{i}{i\not\partial - m + i\varepsilon} \delta^4(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not p - m + i\varepsilon} e^{-ip(x-y)}$$

Which can be simplified using

$$(\not{p} - m)(\not{p} + m) = p^2 - m^2$$

$$\Rightarrow \quad \frac{1}{\not{p} - m + i\varepsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

Interactions

Yukawa coupling

Yukawa theory: $\mathcal{L}_{int} = g\varphi\bar{\psi}\psi$ w/ φ real scalar field, ψ Dirac fermion

$$Z[\bar{\eta}, \eta, J] \propto \exp\left(ig\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\underbrace{\left(i\frac{\delta}{\delta \eta_\alpha(x)}\right)}\left(\frac{1}{i}\frac{\delta}{\delta \bar{\eta}_\alpha(x)}\right)\right) Z_0[\bar{\eta}, \eta, J]$$

Note minus sign. Here to track

$$\frac{\delta}{\delta \eta(x)}\int d^4y\left[\bar{\eta}(x)\psi(y) + \bar{\psi}(y)\eta(x)\right] = -\bar{\psi}(x)$$

vs. $$\frac{\delta}{\delta \bar{\eta}(x)}\int d^4y\left[\bar{\eta}(x)\psi(y) + \bar{\psi}(y)\eta(x)\right] = +\psi(x)$$

w/ $$Z_0[\bar{\eta}, \eta, J] = \exp\left[i\int d^4x\, d^4y\ \bar{\eta}(x) S(x-y)\eta(y)\right]$$

$$\times \exp\left[\frac{i}{2}\int d^4x\, d^4y\ J(x)\,\Delta(x-y)\,J(y)\right]$$

w/ $$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon}e^{-ip(x-y)} \quad \& \quad \Delta(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2 - m_\varphi^2 + i\varepsilon}$$

Want to compute $W(\bar{\eta}, \eta, J)$, the connected Feynman diags. $Z = \exp(iW)$

Note: every interaction involves $\psi + \bar{\psi} \Rightarrow$ define arrows so that every vertex has one arrow flowing in and one out

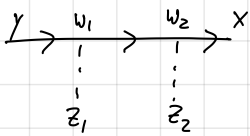
We will do two examples: $e^- \varphi \to e^- \varphi$ and $e^+ e^- \to e^+ e^-$

$e^- \varphi \to e^- \varphi$ derives from LSZ reducing

$$\langle \Omega | T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\varphi(z_1)\varphi(z_2)\}|\Omega\rangle_c = \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\, i\frac{\delta}{\delta\eta_\beta(y)}\,\frac{1}{i}\frac{\delta}{\delta J(z_1)}\,\frac{1}{i}\frac{\delta}{\delta J(z_2)}\, iW\Big|_{\eta=\bar{\eta}=J=0}$$

$$= \left(\frac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \left[S(x-w_2)\,S(w_2-w_1)\,S(w_1-y)\right]_{\alpha\beta}$$

$$\times \Delta(z_1-w_1)\,\Delta(z_2-w_2) + \underbrace{(z_1 \leftrightarrow z_2)}_{\text{identical } \varphi\text{'s}} + \mathcal{O}(g^4)$$



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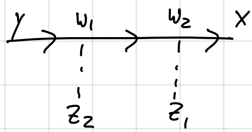


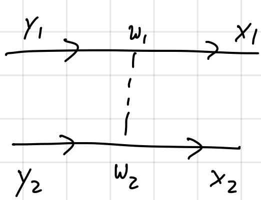
For $e^+e^- \to e^+e^- \Rightarrow \langle\Omega|T\{\psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_1}(y_1)\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_2}(y_2)|\Omega\rangle_c$

$$= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\alpha_1}(x_1)}\, i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\,\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_{\alpha_2}(x_2)}\, i\frac{\delta}{\delta\eta_{\beta_2}(y_2)}\, iW\Big|_{\eta=\bar{\eta}=J=0}$$

$$= \left(\frac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2 \left[S(x_1-w_1)\,S(w_1-y_1)\right]_{\alpha_1\beta_1} \times \Delta(w_1-w_2)$$

$$\times\left[S(x_2-w_2)\,S(w_2-y_2)\right]_{\alpha_2\beta_2} - \left((y_1,\beta_1)\leftrightarrow(y_2,\beta_2)\right) + \mathcal{O}(g^4)$$

where "$-$" is from anti-commuting the $\frac{\delta}{\delta\eta}$ factors.



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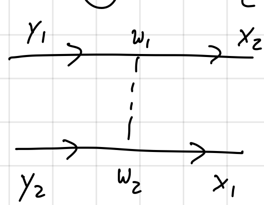


Minus sign for fermion loops

One can derive this fact by carefully evaluating the path integral in perturbation theory.

Here is a heuristic argument:

Consider
$$1 = \frac{\det(\Box + \lambda A(x))}{\det(\Box + \lambda A(x))}$$

We can interpret this as the path integral over a complex bosonic scalar φ and a complex Grassman scalar ψ:

$$1 = \int \mathcal{D}\varphi\, \mathcal{D}\varphi^*\, \mathcal{D}\psi\, \mathcal{D}\psi^*\; e^{i\int d^4x\, \mathcal{L}}$$

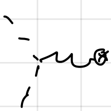
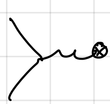
w/ $\quad \mathcal{L} = \varphi^*(\Box + \lambda A(x))\varphi + \psi^*(\Box + \lambda A(x))\psi$

and we can think of $A(x)$ as a background classical field.

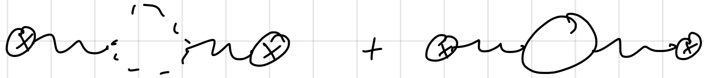
Then we have Feynman rules:

φ: - - - - $= \dfrac{i}{p^2}$

ψ: —— $= \dfrac{i}{p^2}$

 $= i\lambda$  $= i\lambda$

<u>But</u> we know the corrections are trivial since our path integral is equal to 1.

$\Rightarrow$  $= 0$

$\Rightarrow$ Fermionic loops must have opposite sign compared to bosonic loops.